ON SINGLE-VALUED INTEGRALS IN THE PROBLEM OF MOTION OF A SOLID BODY IN A NEWTONIAN FIELD OF FORCES

(OB ODNOZNACHNYKH INTEGRALAKH V ZADACHE O DVIZHENII TVERDOGO TELA V N'IUTONOVSKOM POLE SIL)

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1. As is known [1], the approximate equations of motion of a solid body about a stationary point under the influence of a central Newtonian field of forces, on the premise that the stationary point of the body is located at a sufficiently large distance R from the center of gravity, has the following form, accurate to the second order of magnitude

$$
\begin{aligned}
A\frac{dp}{dt} + (C-B)\,qr &= Mg\,(y_0\gamma'' - z_0\gamma') + \frac{3g}{R}(C-B)\,\gamma'\gamma'' \\
B\frac{dq}{dt} + (A-C)\,rp &= Mg\,(z_0\gamma - x_0\gamma'') + \frac{3g}{R}(A-C)\,\gamma''\gamma \qquad (1)\\
C\frac{dr}{dt} + (B-A)\,pq &= Mg\,(x_0\gamma' - y_0\gamma) + \frac{3g}{R}(B-A)\,\gamma'\gamma \\
\frac{d\gamma}{dt} = r\gamma' - q\gamma'', \qquad \frac{d\gamma'}{dt} &= p\gamma'' - r\gamma, \qquad \frac{d\gamma''}{dt} = q\gamma - p\gamma'
\end{aligned}
$$

One can apply to Equations (1), which generalize the equations of a classical problem of rotation of the solid body about a stationary point, the method of last multiplier, devised by Jacobi. Subsequently, the problem of integrating the equations is reduced to quadratures if four independent integrals which do not contain time, are known.

In a general case the system of Equations (1) has three independent first integrals:

energy integral

$$Ap^2 + Bq^2 + Cr^2 - 2Mg\,(x_0\gamma + y_0\gamma' + z_0\gamma'') + \frac{3g}{R}(A\gamma^2 + B\gamma'^2 + C\gamma''^2) = \text{const}$$

momentum integral

$$Ap\gamma + Bq\gamma' + Cr\gamma'' = \text{const}$$

trivial integral

$$\gamma^2 + \gamma'^2 + \gamma''^2 = 1$$

The fourth integral was found in the following two cases:

(1) When the body is fixed at the center of mass

$$x_0 = y_0 = z_0 = 0$$

(analog to the Euler case in a classical problem of the motion of a solid body about a fixed point) the fourth integral, i.e. the integral of kinetic momentum has the form

$$A^2p^2 + B^2q^2 + C^2r^2 - \frac{3g}{R}(BC\gamma^2 + AC\gamma'^2 + AB\gamma''^2) = \text{const}$$

This integral was first found by de Brun [2] and the problem was reduced to quadratures by Kobb [3] and then in a different way by Kharlamova [4].

(2) When the body is characterized by kinetic symmetry about one of the main axes of inertia and the center of gravity is located on that axis

$$A = B, \qquad x_0 = y_0 = 0$$

(analog of Lagrange case in the classical problem of rotation of a solid body about a fixed point) the fourth integral is

$$r = \text{const}$$

In this case the problem was reduced to quadratures by Beletskii [1]. Thus, it was possible to integrate Equations (1) only in the above two cases.

2. If one considers time as a complex variable, then for integration of the system (1) a special interest attaches to that case in which the integrals of the system have only poles as their movable singular points. In those cases it is possible to obtain complete solution of the problem by means of setting up differential equations for functions, the relation of which would give in accordance with the theorem of Weierstrass the meromorphic integrals of system (1) and their integration using power series [5]. As a first step in the solution of this problem it is necessary to find all those cases when the system (1) has general integrals which represent single-valued functions of time.

In this paper it is shown that finding all cases in which the integrals of differential Equations (1) are single-valued does not lead to new cases but reduces to solutions for two aforementioned cases. To prove this statement we use the method of a small parameter according to Golubev [6].

By means of substitutions

$$p = \alpha p_1, \quad q = \alpha q_1, \quad r = \alpha r_1, \quad \gamma = \alpha^2\gamma_1, \quad \gamma' = \alpha^2\gamma_1', \quad \gamma'' = \alpha^2\gamma_1'', \quad t = t_0 + \frac{t_1}{\alpha}$$

the system of Equations (1) is transformed into the system

$$\begin{aligned}
&A\,\frac{dp_1}{dt_1} + (C-B)\,q_1r_1 = Mg\,(y_0\gamma_1'' - z_0\gamma_1') + \alpha^2\,\frac{3g}{R}\,(C-B)\,\gamma_1'\gamma_1'' \\
&B\,\frac{dq_1}{dt_1} + (A-C)\,r_1p_1 = Mg\,(z_0\gamma_1 - x_0\gamma_1'') + \alpha^2\,\frac{3g}{R}\,(A-C)\,\gamma_1''\gamma_1 \\
&C\,\frac{dr_1}{dt_1} + (B-A)\,p_1q_1 = Mg\,(x_0\gamma'_1 - y_0\gamma_1) + \alpha^2\,\frac{3g}{R}\,(B-A)\,\gamma_1\gamma_1' \\
&\frac{d\gamma_1}{dt_1} = r_1\gamma_1' - q_1\gamma_1'', \qquad \frac{d\gamma_1'}{dt_1} = p_1\gamma_1' - r_1\gamma_1, \qquad \frac{d\gamma_1''}{dt_1} = q_1\gamma_1 - p_1\gamma_1'
\end{aligned} \tag{2}$$

which contains a small parameter α^2. For $\alpha = 0$ we obtain a system of simplified equations which represent the classical equations of motion of a heavy solid body about a fixed point [7]. Concerning these equations it is known [8] that they have single-valued integrals only in three cases:

$$\begin{aligned}
&1) \quad x_0 = y_0 = z_0 = 0 \\
&2) \quad A = B, \quad x_0 = y_0 = 0 \\
&3) \quad A = B = 2C, \quad y_0 = z_0 = 0
\end{aligned}$$

Therefore, in order to prove the above statement it is sufficient to show that the system (1) rewritten for the third of the above cases in the form

$$\begin{aligned}
&\frac{dp}{dt} - \frac{1}{2}\,qr = -b^2\gamma'\gamma'', && \frac{d\gamma}{dt} = r\gamma' - q\gamma'' \\
&\frac{dq}{dt} + \frac{1}{2}\,pr = -\frac{c}{2}\,\gamma'' + b^2\gamma\gamma'', && \frac{d\gamma'}{dt} = p\gamma'' - r\gamma \\
&\frac{dr}{dt} = c\gamma', && \frac{d\gamma''}{dt} = q\gamma - p\gamma'
\end{aligned} \tag{3}$$

$$\left(b^2 = \frac{3g}{2R}, \quad c = \frac{Mgx_0}{C}\right)$$

has nonsingle-valued integrals.

We introduce a small parameter α in accordance with

$$p = \frac{p_1}{\alpha}, \; q = \frac{q_1}{\alpha}, \; r = \frac{r_1}{\alpha}, \; \gamma = \frac{\gamma_1}{\alpha}, \; \gamma' = \frac{\gamma_1'}{\alpha}, \; \gamma'' = \frac{\gamma_1''}{\alpha}, \; t = t_0 + \alpha t_1$$

Then, Equations (3) have the form

$$\begin{aligned} &\frac{dp_1}{dt_1} - \frac{1}{2} q_1 r_1 = -b^2 \gamma_1' \gamma_1'', && \frac{d\gamma_1}{dt_1} = r_1 \gamma_1' - q_1 \gamma_1'' \\ &\frac{dq_1}{dt_1} + \frac{1}{2} p_1 r_1 = b^2 \gamma_1'' \gamma_1 - \alpha \frac{c}{2} \gamma_1'', && \frac{d\gamma_1'}{dt_1} = p_1 \gamma_1'' - r_1 \gamma_1 \\ &\frac{dr_1}{dt_1} = \alpha c \gamma_1', && \frac{d\gamma_1''}{dt_1} = q_1 \gamma_1 - p_1 \gamma_1' \end{aligned} \qquad (4)$$

For $\alpha = 0$ we obtain a system of simplified equations

$$\begin{aligned} &\frac{dp_1}{dt_1} - \frac{1}{2} q_1 r_1 = -b^2 \gamma_1' \gamma_1'', && \frac{d\gamma_1}{dt_1} = r_1 \gamma_1' - q_1 \gamma_1'' \\ &\frac{dq_1}{dt_1} + \frac{1}{2} p_1 r_1 = b^2 \gamma_1'' \gamma_1, && \frac{d\gamma_1'}{dt_1} = p_1 \gamma_1'' - r_1 \gamma_1 \\ &\frac{dr_1}{dt_1} = 0, && \frac{d\gamma_1''}{dt_1} = q_1 \gamma_1 - p_1 \gamma_1' \end{aligned} \qquad (5)$$

This system is fulfilled by the following partial integrals

$$p_1 = \frac{i}{t_1}, \quad q_1 = r_1 = \gamma_1 = 0, \quad \gamma_1' = \frac{1}{bt_1}, \quad \gamma_1'' = \frac{i}{bt_1} \qquad (i = \sqrt{-1})$$

Expanding integrals of the system (4) with respect to the parameter α

$$p_1 = \frac{i}{t_1} + \alpha p_2 + \ldots, \quad q_1 = \alpha q_2 + \ldots, \quad r_1 = \alpha r_2 + \ldots, \quad \gamma_1 = \alpha \gamma_2 + \ldots \qquad (6)$$

$$\gamma_1' = \frac{1}{bt_1} + \alpha \gamma_2' + \ldots \qquad \gamma_1'' = \frac{i}{bt_1} + \alpha \gamma_2'' + \ldots$$

and then substituting into that system the above expansions (6) and subsequently equating the coefficients for α, we obtain equations which determine p_2, q_2, r_2, γ_2, γ_2', γ_2'', for example

$$\frac{dr_2}{dt_1} = \frac{c}{bt_1} \qquad (7)$$

Inasmuch as the quantity c/b is different from zero it follows from (7) that there is a critical variable logarithmic point in r_2. Thus, the system (3) has multivalued integrals which proves the above advanced statement.

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